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Structural aspects of functional integral bosonization

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Abstract. We discuss structural aspects of functional integral bosonization of two-dimensional models. We show that the use of auxiliary vector fields enlarges the Hilbert space by the introduction of an external field algebra that should not be considered as an element of the intrinsic algebraic structure defining the model. These aspects are discussed in a model with well known and established results in the literature, by considering the Abelian reduction of the Wess–Zumino–Witten theory to reconstruct in the Hilbert space of states Coleman's proof of the fermion–boson mapping between the massive Thirring and sine–Gordon theories. We show that the factorization of the partition function will generally lead to incorrect conclusions concerning the physical content of the model, such as the existence of infinitely delocalized states and the violation of the asymptotic factorization property of the Wightman functions. In order to exert control on the effect of the redundant Bose fields and obtain the fermion–boson mapping in the Hilbert space of states, the functional integral bosonization must be performed on the generating functional.

1. Introduction

Over the last few years, an impressive effort has been made by many physicists to understand the underlying properties of quantum field theories in two dimensions [1], as well as to try to picture these models as theoretical laboratories to obtain insight into more realistic four-dimensional field theories and, more recently, to apply them to low-dimensional condensed matter systems [2], as well as to N-body problems in nuclear physics [3, 4]. After over a quarter of a century of investigations on two-dimensional field theories we have learned that, besides their peculiar formal aspects, two-dimensional models also have the value of providing a better conceptual and structural understanding of general properties of quantum field theory [4–7].

In the recent efforts towards the extension of the bosonization procedure to 2+1 dimensions [8], use has been made of an interpolating field procedure that leads to a 'mapping' of the partition function of the original theory into a partition function of Chern–Simons-type theories. At the present state of the research, and due to the large number of papers on the subject, it seems to be very instructive to make a foundational investigation of the basic structural properties of the functional integral bosonization in order to ensure the correct mathematical and physical interpretation of the established fermion–boson mappings.

In this paper we shall discuss some structural aspects of functional integral bosonization of two-dimensional Abelian models. In [9, 11] two-dimensional models with quartic Fermi field interaction have been analysed within the path-integral framework. In [10] the functional

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integral version of Coleman's proof of the 'equivalence' between the massive Thirring and sine–Gordon theories is presented. The bosonization scheme introduced in [9] requires the introduction of auxiliary vector fields in the functional integral in order to recast the theory in terms of a Lagrangian which is quadratic in the Fermi fields. This procedure has also been applied to the bosonization of the Abelian and non-Abelian Thirring models [8] in 2 + 1 dimensions. As stressed in [12, 13], the bosonization procedure introduces a redundant Bose field algebra which contains more degrees of freedom than those needed for the description of the physical content of the model. However, the structural aspects related to the appearance of decoupled massless Bose fields in the functional integral bosonization have not been fully appreciated and clarified in the preceding literature. We shall discuss the role played by the auxiliary vector field, as well as by the 'decoupled' massless scalar fields in the functional integral bosonization, which are in general obscure in the existing literature on this subject, since it is common practice [10, 4] to discard these decoupled fields through the bosonization of the partition function. From our point of view, and in agreement with the procedure adopted in [9], the most appropriate way to treat the problem is to perform the bosonization of the generating functional of the theory, from which we construct the Hilbert space of the model, without disregarding the role played by the decoupled massless Bose fields. As was shown in [13, 14], by relaxing the control on the construction of the Hilbert space of two-dimensional anomalous chiral models, some misleading conclusions on the physical content of the model can arise, such as for instance, the θ-vacuum representation and the suggested equivalence of the vector Schwinger model and the chiral Schwinger model defined for the regularization-dependent parameter $a = 2$ [15]. Within the same approach, the chiral QCD_2 was discussed in [14] and it was shown that a construction based on a redundant field algebra will generally lead to incorrect conclusions concerning its physical properties, such as the equivalence of chiral and vector QCD_2 and the existence of an infinite degeneracy of the ground state in the chiral $U(N)$ models.

In order to discuss these subtle aspects involved in the functional integral bosonization, in this paper we shall consider the well known two-dimensional massive Thirring model. To this end, we review the presentation of [10] by using the Abelian reduction of the Wess–Zumino–Witten (WZW) theory [13, 17, 18] to reconstruct in the Hilbert space of states Coleman's proof of the fermion–boson mapping between the massive Thirring and sine–Gordon theories [16]. The redundant decoupled massless Bose fields are kept through the bosonization of the generating functional of the theory. We show that their only effect is to generate constant contributions to the Wightman functions in the Hilbert space of states. The original Fermi field of the massive Thirring model is bosonized in terms of the 'soliton' Mandelstam field and a spurious exponential operator with zero scale dimension. In contrast to what occurs in two-dimensional gauge theories [20, 21], this spurious field has no physical consequences and reduces to the identity in the Hilbert space of states. In the present approach close attention is paid to maintaining complete control on the Hilbert space structure needed for the representation of the intrinsic field algebra generated by the set of fundamental fields whose Wightman functions define the model. We show that the factorization of the partition function of the effective bosonized theory leads to incorrect conclusions concerning the physical content of the model, such as the existence of infinitely delocalized states and the violation of the asymptotic factorization property of the Wightman functions (cluster decomposition property). The present approach clarifies some delicate mathematical aspects which are not evident in the presentation of [10, 11] and also streamlines the discussion of the massless Thirring model presented in [9].

2. Functional integral bosonization

To begin with, consider the two-dimensional Abelian massive Thirring model [16] defined by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m_o)\psi - \tfrac{1}{2}g^2 J^\mu J_\mu \tag{2.1}$$

where J^μ is the fermionic current†, $J^\mu = \bar{\psi}\gamma^\mu\psi$.

Within the operator formulation the Hilbert space $\mathcal{H}$ of the model is constructed as a representation of the intrinsic field algebra $\Im$ generated by the set of fundamental local field operators $\{\bar{\psi}, \psi\}$, and whose Wightman functions define the theory, $\mathcal{H} \doteq \Im\{\bar{\psi}, \psi\}|0\rangle$. The composite operators belonging to the polynomial field algebra $\Im$ are the bilocals $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$, since only gauge invariance of the first kind is required.

In the functional integral formalism, the Hilbert space of the massive Thirring model can be built from the generating functional with source terms for the basic fields that generate the polynomial field algebra $\Im$, i.e.

$$\mathcal{Z}[\theta, \bar{\theta}] = \mathcal{N}^{-1} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathrm{e}^{\mathrm{i}\mathbb{W}[\bar{\psi},\psi,\theta,\bar{\theta}]} \tag{2.2}$$

where $\mathbb{W}[\bar{\psi}, \psi, \theta, \bar{\theta}]$ is the action in the presence of external Grassmann-valued sources θ and $\bar{\theta}$,

$$\mathbb{W}[\bar{\psi}, \psi, \theta, \bar{\theta}] = \int \mathrm{d}^2x\, \{\mathcal{L} + \bar{\psi}\theta + \bar{\theta}\psi\}. \tag{2.3}$$

Following the procedure adopted in [8–10], as a first step in the bosonization of the model we define an enlarged field algebra $\Im'$ by introducing an 'auxiliary' vector field a_μ, such that $\Im' = \Im'\{\bar{\psi}, \psi, a_\mu\}$, and considering the 'interpolating' generating functional

$$\mathcal{Z}'[\theta, \bar{\theta}, \zeta_\mu] = \mathcal{N}^{-1} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \int \mathcal{D}a_\mu\, \exp\left\{\mathrm{i}\int \mathrm{d}^2x\, \{\tfrac{1}{2}a^\mu a_\mu + a_\mu\zeta^\mu\}\right\} \mathrm{e}^{\mathrm{i}\mathbb{W}[\bar{\psi},\psi,\theta,\bar{\theta}]}. \tag{2.4}$$

The source term for the auxiliary vector field a_μ was included in order to control the effects of the bosonization on the construction of the Hilbert space $\mathcal{H}' \doteq \Im'\{\bar{\psi}, \psi, a_\mu\}|0\rangle$. As we shall see, the bosonized generating functional $\mathcal{Z}'$ defines an enlarged positive semi-definite Hilbert space.

The next step in the functional bosonization is to reduce the action of the Thirring model to a quadratic action in the Fermi field by performing the 'change of variables' [8, 9]

$$a_\mu = \mathcal{A}_\mu - gJ_\mu \tag{2.5}$$

such that

$$\int \mathcal{D}a_\mu\, \exp\left\{\mathrm{i}\int \mathrm{d}^2x\, \tfrac{1}{2}\{a^\mu a_\mu - g^2 J^\mu J_\mu\}\right\} = \int \mathcal{D}\mathcal{A}_\mu\, \exp\left\{\mathrm{i}\int \mathrm{d}^2x\, \{\tfrac{1}{2}\mathcal{A}_\mu\mathcal{A}^\mu - gJ^\mu\mathcal{A}_\mu\}\right\}. \tag{2.6}$$

† Our conventions are: $x^\pm = x^0 \pm x^1$; $\partial_\pm = \partial_0 \pm \partial_1$; $\mathcal{A}^\pm = \mathcal{A}^0 \pm \mathcal{A}^1$;

$$g^{00} = 1 = -g^{11} \qquad \epsilon^{01} = -\epsilon^{10} = 1 \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^5 = \gamma^0\gamma^1 \qquad \gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu.$$

The scalar and pseudoscalar massless free fields are decomposed as $\phi(x) = \phi(x^+) + \phi(x^-)$, $\tilde{\phi}(x) = \phi(x^+) - \phi(x^-)$, such that $\partial_\mu\phi(x) = \epsilon_{\mu\nu}\partial^\nu\tilde{\phi}(x)$.

This leads to a 'new' effective action in (2.4) which is given in terms of the Lagrangian density

$$\mathcal{L}_{eff} = \bar{\psi}\not{D}(\mathcal{A})\psi - m_o\bar{\psi}\psi + \tfrac{1}{2}\mathcal{A}_\mu\mathcal{A}^\mu \tag{2.7}$$

where the covariant derivative is defined by $\not{D}(\mathcal{A}) \doteq (\mathrm{i}\not{\partial} - g\not{\mathcal{A}})$. The fermionic piece of the effective theory exhibits local gauge invariance. The local gauge non-invariance of the model is carried by the last term in the effective Lagrangian density (2.7). In order to decouple the Fermi and vector fields in the Lagrangian (2.7), we introduce the parametrization of the vector field components $\mathcal{A}_\pm$ in terms of the $U(1)$ group-valued Bose fields (U, V) as [14]

$$\mathcal{A}_+ = \frac{1}{g}U^{-1}\mathrm{i}\partial_+ U \qquad \mathcal{A}_- = \frac{1}{g}V\mathrm{i}\partial_- V^{-1} \tag{2.8}$$

such that

$$\bar{\psi}\not{D}(\mathcal{A})\psi = (V^{-1}\psi_{(1)})^\dagger(\mathrm{i}\partial_-)(V^{-1}\psi_{(1)}) + (U\psi_{(2)})^\dagger(\mathrm{i}\partial_+)(U\psi_{(2)}). \tag{2.9}$$

The decoupling is performed by the (Abelian) fermion chiral rotation [13]

$$\psi = \Omega\chi \tag{2.10}$$

where the chiral rotation matrix Ω is given by

$$\Omega = \tfrac{1}{2}(1+\gamma^5)U^{-1} + \tfrac{1}{2}(1-\gamma^5)V. \tag{2.11}$$

Introducing in the functional integral the identities

$$\begin{aligned} 1 &= \int \mathcal{D}U\,[\det\mathcal{D}_+(U)]\delta(g\mathcal{A}_+ - U^{-1}\mathrm{i}\partial_+ U) \\ 1 &= \int \mathcal{D}V\,[\det\mathcal{D}_-(V)]\delta(g\mathcal{A}_- - V\mathrm{i}\partial_- V^{-1}) \end{aligned} \tag{2.12}$$

the change of variables from $(\mathcal{A}_+, \mathcal{A}_-)$ to (U, V) is performed by integrating over the vector field components $\mathcal{A}_\pm$. Performing the fermion chiral rotation (2.10) and taking into account the corresponding change in the integration measure [13], we obtain

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\mathcal{A}_+\,\mathcal{D}\mathcal{A}_- = \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathcal{D}U\,\mathcal{D}V\,\mathcal{J}[U,V] \tag{2.13}$$

with

$$\mathcal{J}[U,V] = \exp\left\{-\mathrm{i}(\Gamma[U]+\Gamma[V]) - \mathrm{i}\frac{c}{g^2}\int \mathrm{d}^2z\,(U^{-1}\partial_+U)(V\partial_-V^{-1})\right\} \tag{2.14}$$

where $\Gamma[G]$ is the Wess–Zumino–Witten functional [17], which enters in (2.14) with a negative level. In the Abelian case the WZW functional reduces to the free action

$$\Gamma[G] = \Gamma[G^{-1}] = \frac{1}{8\pi}\int \mathrm{d}^2z\,\partial_\mu G^{-1}\partial^\mu G. \tag{2.15}$$

Due to the absence of local gauge invariance, the last term in (2.14) has been added by exploiting the regularization freedom in the computation of the Jacobians. Since the effective fermionic theory is invariant under local gauge transformations we shall use the 'gauge-invariant regularization' by setting† $c = g^2/4\pi$.

† A different choice of regularization implies a redefinition of the β parameter of the sine–Gordon theory (equation (2.31)) and the physical range for the coupling constant g. Defining the arbitrary regularization parameter $\boldsymbol{a}$ by $c = g^2\boldsymbol{a}/4\pi$, we obtain $\beta^2 = 4\pi[1+(g^2/2\pi)(\boldsymbol{a}-1)]/[1+(g^2/2\pi)(\boldsymbol{a}+1)]$. This also leads to unconventionally normalized currents. The standard theory corresponds to setting $\boldsymbol{a} = 1$.

Using the Polyakov–Wiegman identity [18]

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\int d^2z\, \big(U^{-1}\partial_+ U\big)\big(V\partial_- V^{-1}\big) \tag{2.16}$$

we obtain

$$\mathcal{J} = e^{-i\Gamma[UV]} \tag{2.17}$$

where $\Sigma \doteq UV$ is a gauge-invariant field. The effective action is given by

$$\mathbb{W}_{eff} = \int d^2z\, \big\{\bar\chi i\partial\!\!\!/ \chi - m_o\big(\chi^*_{(1)}\chi_{(2)}(UV)^{-1} + \chi^*_{(2)}\chi_{(1)}(UV)\big)\big\} \\ -\Gamma[UV] - \frac{1}{2g^2}\int d^2x\, \big(U^{-1}\partial_+ U\big)\big(V\partial_- V^{-1}\big). \tag{2.18}$$

The vector fields in two dimensions can be decomposed as

$$\mathcal{A}_\mu = -\frac{1}{g}(\epsilon_{\mu\nu}\,\partial^\nu\,\tilde\phi + \partial_\mu\eta) \tag{2.19}$$

which corresponds to parametrizing the Bose fields (U, V) as follows:

$$U = e^{i(\tilde\phi+\eta)} \qquad V = e^{i(\tilde\phi-\eta)}. \tag{2.20}$$

The effective Lagrangian density, corresponding to the action (2.18), can be written as

$$\mathcal{L}_{eff} = \mathcal{L}_I[\bar\chi, \chi, \tilde\phi'] + \frac{1}{2g^2}(\partial_\mu\eta)^2 \tag{2.21}$$

where

$$\mathcal{L}_I[\bar\chi, \chi, \tilde\phi'] = -\tfrac{1}{2}(\partial_\mu\tilde\phi')^2 + \bar\chi i\partial\!\!\!/\chi - m_o(\chi^*_{(1)}\chi_{(2)}e^{-2i\alpha\tilde\phi'} + \text{h.c.}) \tag{2.22}$$

with

$$\tilde\phi' = \left(\frac{1+g^2/\pi}{g^2}\right)^{1/2}\tilde\phi = \alpha^{-1}\tilde\phi. \tag{2.23}$$

The $2n$-point functions of the Fermi fields of the massive Thirring model are obtained by functional derivation of the generating functional with respect to the Grassmann-valued sources $\bar\theta$ and θ, and can be written as

$$\langle 0|\bar\psi(x_1)\ldots\bar\psi(x_n)\psi(y_1)\ldots\psi(y_n)|0\rangle' = \langle 0|\prod_{i=1}^{n} e^{i\eta(x_i)}\prod_{j=1}^{n} e^{-i\eta(y_j)}|0\rangle_o \\ \times\langle 0|\prod_{i=1}^{n}\bar\chi(x_i)\,e^{-i\alpha\gamma^5\tilde\phi'(x_i)}\prod_{j=1}^{n}\chi(y_j)\,e^{i\alpha\gamma^5\tilde\phi'(y_j)}|0\rangle_I \tag{2.24}$$

where the notation $\langle 0|\cdot|0\rangle_o$ denotes an average with respect to the free massless η-field theory and $\langle 0|\cdot|0\rangle_I$ denotes an average with respect to the effective Lagrangian $\mathcal{L}_I[\bar\chi, \chi, \tilde\phi']$. The computation of the correlation functions (2.24) is performed using the vacuum functional of the theory, which provides the functional integral Gell'Mann–Low formula. Following the standard procedure [6, 10], we perform the expansion of the exponential of the interaction term of $\mathcal{L}_I[\bar\chi, \chi, \tilde\phi']$ in a power series of the bare mass m_o. The resulting correlation function, besides the contributions of the field $\tilde\phi'$, corresponds to averages of products of chiral density operators $\bar{\chi}(x_i)\chi(y_j)$ with respect to the free Fermi theory. The effective bosonized theory can be obtained by reconstructing the series using the free-field bosonization expressions†

$$\chi(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi\gamma^5} : \exp\left\{\mathrm{i}\sqrt{\pi}\left\{\gamma^5\tilde{\varphi}(x) + \int_{x^1}^{+\infty} \dot{\tilde{\varphi}}(x^0, z^1)\,\mathrm{d}z^1\right\}\right\} : \tag{2.25}$$

$$\bar{\chi}\mathrm{i}\partial\!\!\!/\chi \equiv \tfrac{1}{2} : (\partial_\mu\tilde{\varphi})^2 : \tag{2.26}$$

$$\chi^*_{(1)}\chi_{(2)} \equiv \frac{\mu_o}{2\pi} : \mathrm{e}^{2\mathrm{i}\sqrt{\pi}\tilde{\varphi}} : \tag{2.27}$$

where $:\cdot:$ indicates normal ordering with respect to the free propagator $(\Box + \mu_o^2)^{-1}$ in the limit $\mu_o \to 0$. In this way, the effective bosonized Lagrangian density is given by

$$\mathcal{L}^{bos}_{eff} = \tfrac{1}{2}(\partial_\mu\tilde{\varphi})^2 - \tfrac{1}{2}(\partial_\mu\tilde{\phi}')^2 + \frac{1}{2g^2}(\partial_\mu\tilde{\eta})^2 - \left(\frac{m_o\mu_o}{\pi}\right)\cos\left\{2\sqrt{\pi}\tilde{\varphi} + 2\alpha\tilde{\phi}'\right\}. \tag{2.28}$$

In order to recast the Lagrangian (2.28) in terms of the standard sine–Gordon theory, we introduce two independent fields $\tilde{\Phi}$ and $\tilde{\xi}$ through the canonical transformation

$$\beta\tilde{\Phi} = 2\sqrt{\pi}\tilde{\varphi} + 2\alpha\tilde{\phi}' \tag{2.29}$$

$$\beta\tilde{\xi} = 2\alpha\tilde{\varphi} + 2\sqrt{\pi}\tilde{\phi}' \tag{2.30}$$

with

$$\beta^2 = \frac{4\pi}{1 + g^2/\pi} \tag{2.31}$$

and the field ξ is quantized with a negative metric. In terms of these new fields, the effective bosonized Lagrangian density can be written as

$$\mathcal{L}^{bos}_{eff} = -\frac{1}{2}(\partial_\mu\tilde{\xi})^2 + \frac{1}{2g^2}(\partial_\mu\tilde{\eta})^2 + \frac{1}{2}(\partial_\mu\tilde{\Phi})^2 - m'_o\cos[\beta\tilde{\Phi}]. \tag{2.32}$$

The effective bosonized theory is described by the sine–Gordon theory and two decoupled massless fields quantized with opposite metrics. As a matter of fact, the extraction of these 'decoupled' massless Bose fields relies on a structural problem which is related to the fact that the fields η and ξ do not belong to the field algebra $\Im'$ and cannot be defined by themselves as operators in the Hilbert space of states [12, 13, 14]. We shall return to this point later.

3. Field algebra and Hilbert space

It is instructive to express the original Fermi field ψ and the source terms in the generating functional (2.4) in terms of the sine–Gordon field $\tilde{\Phi}$. Performing the fermion chiral rotation (2.10), together with the canonical transformations (2.30), and using the bosonized expression (2.25) for the free massive Fermi field, we obtain the Fermi field $\psi(x)$ of the massive Thirring model in terms of the Mandelstam [19] 'soliton' field $\Psi^\Phi(x)$ as

$$\psi_{(\alpha)}(x) = \Omega(x)_{\alpha\beta}\chi_\beta(x) = \Psi^\Phi_{(\alpha)}(x)\,\sigma(x) \tag{3.33}$$

where

$$\Psi^\Phi_{(\alpha)}(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi\gamma^5_{\alpha\alpha}} : \exp\mathrm{i}\left\{\frac{\beta}{2}\gamma^5_{\alpha\alpha}\tilde{\Phi}(x) + \frac{2\pi}{\beta}\int_{x^1}^{+\infty} \dot{\tilde{\Phi}}(x^0, z^1)\,\mathrm{d}z^1\right\} : \tag{3.34}$$

† In order to consider the massive Thirring model as a mass perturbation around the fixed point of the massless theory we must require for the dimension of the mass operator $\dim(\bar{\psi}\psi) = \beta^2/4\pi < 2$, where $\beta^2 = 4\pi/(1 + g^2/\pi)$.

and represents the original Fermi field degrees of freedom [19]. In equation (3.33) both spinor components of the Mandelstam field $\Psi^{\Phi}_{(\alpha)}(x)$ are multiplied by the exponential field

$$\sigma(x) = e^{i[\eta(x)+g\xi(x)]}. \tag{3.35}$$

The auxiliary vector field (2.19) can be written as

$$\mathcal{A}_\mu = -g\frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\tilde{\Phi} + \ell_\mu \tag{3.36}$$

where ℓ_μ is a longitudinal current

$$\ell_\mu = -\partial_\mu\left(\xi + \frac{1}{g}\eta\right) = \partial_\mu\ell. \tag{3.37}$$

Taking into account the invariance of the fermionic part of the effective theory under local gauge transformations, the fermionic current $J^\mu = \bar{\psi}\gamma^\mu\psi$, is computed using a gauge-invariant regularization, and we obtain the bosonized expression [16, 19]

$$J^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\tilde{\Phi}. \tag{3.38}$$

The bosonized interpolating generating functional (2.4) can be written as

$$\mathcal{Z}'[\theta,\bar{\theta},\zeta^\mu] = \mathcal{N}^{-1}\int\mathcal{D}\tilde{\Phi}\,e^{i\mathbb{W}[\tilde{\Phi}]}\int\mathcal{D}\eta\,e^{i\mathbb{W}_o[\eta]}\int\mathcal{D}\xi\,e^{-i\mathbb{W}_o[\xi]}$$
$$\times\exp\left\{i\int d^2x\,\{(\overline{\Psi}^{\Phi}\sigma^*)\,\theta + \bar{\theta}(\Psi^{\Phi}\sigma) + \zeta^\mu\ell_\mu\}\right\} \tag{3.39}$$

where $\mathbb{W}_o[\eta]$ is the free action for the non-canonical massless field η, $\mathbb{W}_o[\xi]$ is the free action for the massless field ξ, which is quantized with negative metric and $\mathbb{W}[\tilde{\Phi}]$ is the action for the sine–Gordon field $\tilde{\Phi}$.

From the generating functional $\mathcal{Z}'$ we obtain the general $2n$-point functions for the Fermi field of the massive Thirring model in terms of averages of order-disorder operators of the sine–Gordon theory

$$\langle 0|\bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(y_1)\ldots\psi(y_n)|0\rangle' = \langle 0|\overline{\Psi}^{\Phi}(x_1)\ldots\overline{\Psi}^{\Phi}(x_n)\,\Psi^{\Phi}(y_1)\ldots\Psi^{\Phi}(y_n)|0\rangle$$
$$\times\langle 0|\sigma^*(x_1)\ldots\sigma^*(x_n)\,\sigma(y_1)\ldots\sigma(y_n)|0\rangle_o \tag{3.40}$$

where the notation $\langle 0|\cdot|0\rangle$ denotes an average with respect to the sine–Gordon theory and $\langle 0|\cdot|0\rangle_o$ denotes an average with respect to the free theories of the massless Bose fields η and ξ. Due to the opposite metric quantization for the fields η and ξ, the functional integration over the field ξ cancels those arising from the integration over the field η in such a way that the field σ generates constant contributions to the Wightman functions

$$\langle 0|\sigma^*(x_1)\ldots\sigma^*(x_n)\,\sigma(y_1)\ldots\sigma(y_n)|0\rangle_o = 1 \tag{3.41}$$

implying that

$$\langle 0|\bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(y_1)\ldots\psi(y_n)|0\rangle' = \langle 0|\overline{\Psi}^{\Phi}(x_1)\ldots\overline{\Psi}^{\Phi}(x_n)\,\Psi^{\Phi}(y_1)\ldots\Psi^{\Phi}(y_n)|0\rangle. \tag{3.42}$$

In this way, we obtain the fermion–boson mapping between the massive Thirring and sine–Gordon theories in the Hilbert subspace of states $\mathcal{H}'$. For any global gauge-invariant functional $\mathcal{F}\{\bar{\psi},\psi\}\in\Im$, we obtain the general one-to-one mapping

$$\langle 0|\mathcal{F}\{\bar{\psi},\psi\}|0\rangle' \equiv \langle 0|\mathcal{F}\{\overline{\Psi}^{\Phi},\Psi^{\Phi}\}|0\rangle. \tag{3.43}$$

From the generating functional (3.39) we see that the longitudinal current (3.37) generates zero norm states from the vacuum

$$\langle 0|\ell_\mu(x)\,\ell_\mu(y)|0\rangle = 0 \tag{3.44}$$

and thus the Hilbert space $\mathcal{H}'$ is positive semi-definite.

Although the partition function obtained from (3.39) factorizes in the form

$$\mathcal{Z}'[0] = \mathcal{Z}^o_\eta[0] \times \mathcal{Z}^o_\xi[0] \times \mathcal{Z}_\Phi[0] \tag{3.45}$$

the fact that the spurious field σ appears attached to the bosonized Fermi field Ψ^Φ in the source terms implies that the generating functional (3.39) cannot be factorized and the massless scalar fields cannot be removed in a naive way, contrary to what is usually done [10, 4]. As a matter of fact, the bosonization procedure leads to the appearance of the spurious field

$$\sigma(z) = \exp\{\mathrm{i}g\ell(z)\} \tag{3.46}$$

implying a structural problem that refers to the existence of infinitely delocalized states $\sigma^n|0\rangle$ in $\mathcal{H}'$, and that would imply the violation of the asymptotic factorization property of the corresponding Wightman functions. Although the field σ generates constant contributions to the Wightman functions, this field cannot be defined by itself in $\mathcal{H}'$ and the cluster decomposition property is not violated. This question can be clarified on the basis of the intrinsic algebraic structure of the model.

The set of fields $\{\bar\psi, \psi\}$ constitute the intrinsic mathematical structure of the Thirring model and generates the local polynomial field algebra $\Im = \Im\{\bar\psi, \psi\}$. The Wightman functions generated from the field algebra $\Im$ define the model and identify the Hilbert space $\mathcal{H}$ of the theory, $\mathcal{H} \doteq \Im|0\rangle$. The introduction of the auxiliary vector field a_μ enlarges the field algebra $\Im \to \Im' = \Im'\{a_\mu, \bar\psi, \psi\}$, and the change of variables (2.5) leads to a field algebra $\Im' = \Im'\{A_\mu, \bar\psi, \psi\}$. This field algebra is represented in the enlarged Hilbert space $\mathcal{H}' \doteq \Im'|0\rangle$. Within the bosonization procedure the fundamental fields defining the field algebra $\Im'$ are written in terms of the Bose fields $\{\tilde\Phi, \tilde\eta, \tilde\xi\}$. This set of Bose fields define an enlarged redundant field algebra $\Im^B$, which is represented in the indefinite metric Hilbert space $\mathcal{H}^B \doteq \Im^B|0\rangle$. These Bose fields are the building blocks in terms of which the bosonized solution is constructed and, as stressed in [12, 13, 14], should not be considered as elements of the intrinsic field algebras $\Im'$. Only some particular combinations of them belong to the field algebra $\Im'$, in such a way that, $\Im' \subset \Im^B$, and thus, $\mathcal{H}' \subset \mathcal{H}^B$. The auxiliary vector field $\mathcal{A}_\mu = gJ_\mu + \ell_\mu$, belongs to the field algebra $\Im'$ and since $J_\mu \in \Im'$, then, $\ell_\mu \in \Im'$. In this way, the positive semi-definite Hilbert space $\mathcal{H}'$ is generated from the field algebra $\Im'\{\mathcal{A}_\mu, \bar\psi, \psi\} = \Im'\{\Im_o, \overline{\Psi}^\Phi\sigma^*, \Psi^\Phi\sigma\}$, where $\Im_o \subset \Im'$ is the field subalgebra generated by the longitudinal current ℓ_μ, $\Im_o = \Im_o\{\ell_\mu\}$, and that generates zero norm states: $\mathcal{H}_o \doteq \Im_o|0\rangle \subset \mathcal{H}'$. The field $\ell = -(\xi + (1/g)\eta)$, that acts as the potential for the longitudinal current ℓ_μ, does not belong to the field algebra $\Im'$ and only its spacetime derivatives occur in $\Im'$. In this way, the exponential field σ given by (3.46) also does not belong to $\Im'$. Since the field σ cannot be defined by itself in $\mathcal{H}'$, the Hilbert space cannot be factorized, $\mathcal{H}' \neq \mathcal{H}_\sigma \otimes \mathcal{H}_{\Psi^\Phi}$.

From the algebraic point of view, the fact that the field σ does not belong to the field algebra $\Im'$ and thus cannot be defined as an operator in $\mathcal{H}'$, follows from the charge content of $\mathcal{H}_B$ and $\mathcal{H}'$, since some *topological* charges† become trivialized in going from $\mathcal{H}^B$ to $\mathcal{H}'$ [12].

† These charges are called *topological* in the sense that the corresponding conservation laws are totally unrelated to any Noether symmetry exhibited by the Lagrangian defining the model.

To begin with, consider the following currents belonging to the Bose field algebra $\Im^B$:

$$j_1^\mu \doteq \mathcal{A}^\mu + \partial^\mu \xi \tag{3.47}$$

$$j_2^\mu \doteq \mathcal{A}^\mu + \frac{1}{g}\partial^\mu \eta. \tag{3.48}$$

Although the vector field $\mathcal{A}_\mu$ belongs to the field algebra $\Im'$, the field derivatives $\partial_\mu \eta$ and $\partial_\mu \xi$ belongs to the Bose field algebra $\Im^B$, they only occur in $\Im'$ as the combination $\ell_\mu = -\partial_\mu(\xi + (1/g)\eta)$. This ensures that $j_i^\mu \in \Im^B$. The corresponding charges are

$$\mathcal{Q}_i \doteq \int_{-\infty}^{+\infty} \mathrm{d}z^1 \, j_i^0(z) \tag{3.49}$$

such that

$$\left[\mathcal{Q}_i, \Im^B\right] \neq 0. \tag{3.50}$$

This implies that the charges $\mathcal{Q}_i$ do not vanish on $\mathcal{H}^B$:

$$\mathcal{Q}_i \mathcal{H}^B \neq 0. \tag{3.51}$$

The charges $\mathcal{Q}_i$ commute with ψ, J^μ and ℓ_μ, that is

$$[\mathcal{Q}_i, \Im_o] = 0 \qquad [\mathcal{Q}_i, \Im] = 0 \quad \rightarrow \quad \left[\mathcal{Q}_i, \Im'\right] = 0. \tag{3.52}$$

This means that the charges $\mathcal{Q}_i$ are trivialized in the restriction from $\mathcal{H}^B$ to $\mathcal{H}'$ [12, 13, 14]:

$$\mathcal{Q}_i \, \mathcal{H}^B \neq 0 \qquad \mathcal{Q}_i \, \mathcal{H}' = 0 \qquad \mathcal{Q}_i \, \mathcal{H} = 0. \tag{3.53}$$

Since $[\mathcal{Q}_i, \sigma] = \alpha\sigma$, the state $|\sigma\rangle = \sigma|0\rangle$ cannot belong to $\mathcal{H}'$ and the field σ cannot be defined as an operator in the Hilbert space $\mathcal{H}'$ [12, 13, 14]. This ensures that the asymptotic factorization property of the Wightman functions holds in $\mathcal{H}'$†.

The states in $\mathcal{H}'$ can be accommodated as equivalence classes modulo ℓ_μ, in such a way that the Hilbert space $\mathcal{H}$ is the quotient space

$$\mathcal{H} \sim \frac{\mathcal{H}'}{\mathcal{H}_o}. \tag{3.54}$$

From the operator point of view, the equivalence established by equation (3.42) implies the algebraic isomorphism

$$\Im\{\bar{\psi}, \psi\} \sim \Im''\{\overline{\Psi}^\Phi \sigma^*, \Psi^\Phi \sigma\} \sim \Im\{\overline{\Psi}^\Phi, \Psi^\Phi\} \tag{3.55}$$

where $\Im''\{\overline{\Psi}^\Phi \sigma^*, \Psi^\Phi \sigma\} \subset \Im'\{\Im_o, \overline{\Psi}^\Phi \sigma^*, \Psi^\Phi \sigma\}$. In this sense we obtain the equivalence

$$\mathcal{Z}'[\theta, \bar{\theta}, 0] \sim \mathcal{Z}[\theta, \bar{\theta}] \sim \mathcal{Z}^\Phi[\theta, \bar{\theta}] \tag{3.56}$$

† A distinct situation occurs in the standard Schwinger model [12, 20, 21], in which the Fermi field belonging to the intrinsic field algebra $\Im_S$ is given by $\psi = \mathrm{e}^{\mathrm{i}\sqrt{\pi}\gamma^5\tilde{\Sigma}}\hat{\psi} \in \Im_S$, where $\tilde{\Sigma}$ is a free scalar field of mass $m = e/\sqrt{\pi}$, $\hat{\psi}$ is given in terms of the free Fermi field ψ_o as $\hat{\psi} = \mathrm{e}^{\mathrm{i}\sqrt{\pi}\gamma^5\tilde{\eta}}\psi_o$ and $\tilde{\eta}$ is a free massless field quantized with a negative metric. Since the field $\tilde{\Sigma}$ can be written in terms of the gauge-invariant field $\mathcal{F}_{\mu\nu} \in \Im_S$, i.e. $\tilde{\Sigma} = (1/2m)\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu} \in \Im_S$, implying that $\mathrm{e}^{\mathrm{i}\sqrt{\pi}\gamma^5\tilde{\Sigma}} \in \Im_S$ and thus $\hat{\psi} \in \Im_S$. In this way the field $\hat{\psi}$ can be defined as operator in the Hilbert space $\mathcal{H}_S$. The vector current is given by $J^\mu = \bar{\psi}\gamma^\mu\psi = -(1/\sqrt{\pi})\epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma} + \ell^\mu$, where ℓ_μ is the longitudinal current $\ell_\mu = -(1/\sqrt{\pi})\partial_\mu(\eta + \phi)$ and ϕ is a massless scalar field that acts as the potential for the conserved free fermionic current. The field algebra of the model is generated by Σ, $\hat{\psi}$ and ℓ_μ, i.e. $\Im_S = \Im_S\{\Sigma, \hat{\psi}, \Im_o\}$, where $\Im_o$ is the field subalgebra generated by ℓ^μ. Since $\hat{\psi} \in \Im_S$, we can define the spurious field $\hat{\psi}^*_{(1)}\hat{\psi}_{(2)} = \sigma_1^*\sigma_2 = \exp\{2\mathrm{i}\sqrt{\pi}(\tilde{\eta} + \tilde{\phi})\} \in \Im_S$. The only reason for $\sigma_1^*\sigma_2$ not being the identity operator in $\mathcal{H}_S$ is that it carries the chiral selection rule, implying that the asymptotic factorization property of the Wightman functions is violated.

with

$$\mathcal{Z}^{\Phi}[\theta,\bar{\theta}] = \mathcal{N}^{-1}\int \mathcal{D}\tilde{\Phi}\,\mathrm{e}^{\mathrm{i}\mathbb{W}[\tilde{\Phi}]}\exp\left\{\mathrm{i}\int \mathrm{d}^2x\,\left\{\overline{\Psi}^{\Phi}(x)\,\theta(x)+\bar{\theta}(x)\,\Psi^{\Phi}(x)\right\}\right\} \tag{3.57}$$

and the fermion–boson mapping between the massive and sine–Gordon theories is established in a positive-definite Hilbert space.

4. Conclusions

In conclusion, we have considered the functional integral bosonization using the auxiliary vector field to discuss a model with well known and established results in the literature. The bosonization procedure introduces extra degrees of freedom that should not be considered as elements of the fundamental field algebra defining the model. In order to exert control on the effect of the redundant Bose fields and obtain the fermion–boson mapping in the Hilbert space of states, the functional integral bosonization must be performed on the generating functional. The factorization of the partition function will generally lead to incorrect conclusions concerning the physical content of the model, such as the existence of infinitely delocalized states and the violation of the asymptotic factorization property of the Wightman functions.

The extension of the bosonization procedure to dimensions higher than two has been the subject of intensive efforts over the last few years. Unfortunately, until recently investigations of the fermion–boson mappings in 2 + 1 dimensions were limited to a perturbative analysis and not to exactly solvable theories. Since these mappings generally are established on the level of factorizable partition functions, a foundational investigation of the basic structural properties of the fermion–boson mappings in 2 + 1 dimensions may offer a valuable lesson for the understanding of the underlying physical properties of the higher-dimensional field theory models. A clear understanding of these points seems to us essential in order to ensure that the fermion–boson mapping is established on the Hilbert space of states and thus may offer information about the true physical content of the original theory.

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